Content

Tutorial 3
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BACK

Questions of today

1. (supplement to the proof of Jense formula in the lecture note). Let Ω be a simply connected open subset of $\mathbb{C}$. Suppose $f:\Omega\to\mathbb{C}$ is a nowhere zero holomorphic function, show that there exits a holomorphic function $g:\Omega\to\mathbb{C}$ such that $f=e^g$.($\log f$ exists!)
2. Let $f:\mathbb{C}\to\mathbb{C}$ is an entire function, and let n be an integer. Show that there exits an entire $g:\Omega\to\mathbb{C}$ such that $f=g^n$ if and only if the orders of zeros are divisible by n.
3. Let f be holomorphic in a region which contains $\overline{D_R}$, and let $a_1,a_2,\dots,a_n$ be the nonzero zeroes of f in D_R. If $|z|<R$, show that if f has a zero at $z=0$ with multiplicity m then
$$\log\left|\frac{f^{(m)}(0)}{m!}\right|+m\log R=-\sum_{k=1}^{n}\log\left(\frac{R}{|a_k|}\right)+\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta$$
4. (Poisson integral formula) Let $f=u+iv$ be holomorphic in a region which contains $\overline{D_R}$, and let $z\in D_R$. Show that
$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)f(re^{i\theta})d\theta.$$
Note that by taking real part of the formula, we have
$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)u(re^{i\theta})d\theta.$$
5. (Poisson-Jensen Formula) Let f be holomorphic in a region which contains $\overline{D_R}$, and let $a_1,a_2,\dots,a_n$ be the zeroes of f in D_R. If $|z|<R$, and $f(z)\neq 0$, show that
$$\log|f(z)|=-\sum_{k=1}^{n}\log\left|\frac{R^2-\overline{a_k}z}{R(z-a_k)}\right|+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)\log|f(Re^{i\theta})|d\theta$$

Hints & solutions of today

1. (See page 100 of textbook for more details)First fix a point $a\in\Omega$, and define $g(z)$ to be the integral of $d\log f=\frac{f'dz}{f}$ over a curve from a to z. You then
   1. Check g is well-defined.
   2. Check g is holomorphic.
   3. Check that $f\exp(-g)$ is a constant.
   4. Modify g to make the constant in (iii) become 1.

   Remark: The nonvanishing of f is used to make sure $d\log f$ is well-defined.
2. The "only if part" is obvious. For the "if part", it would be easy if $f\equiv 0$. So we may assume $f\not\equiv 0$. Fix $a\in\mathbb{C}$ with $f(a)\neq 0$. Define g by
$$g(z)=\exp\left(\frac{1}{n}\int_{\gamma_z}\frac{f'(z)dz}{f}\right),$$
where γ_z is a curve from a to z that does not pass through any of the zeroes of f. As in question 1, we
   1. Check g is well-defined outside the zeroes of f. (By using Argument principle this time, note that the integral without the exponential is not defined)
   2. Check g is holomorphic outside the zeroes of f.
   3. Check that fg^{-n} is a constant outside the zeroes of f.
   4. Modify g to make the constant in (iii) become 1.
   5. Using Riemann extension theorem to show that g in fact extends to the whole $\mathbb{C}$.
3. Write $f=z^m g$ with $g(0)\neq 0$, then apply the Jensen formula for g.
4. We let $w=Re^{i\theta}$, and write out the following
$$\begin{aligned}2\mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)&=\frac{w+z}{w-z}+\frac{\bar w+\bar z}{\bar w-\bar z}\\&=\frac{w+z}{w-z}+\frac{R^2+\bar z w}{R^2-\bar z w}\end{aligned}$$
Note that this expression has a zero at $w=0$, and a simple pole at $w=z$. On the other hand, we have
$$\frac{1}{2\pi}d\theta=\frac{1}{2\pi i}\frac{dw}{w}$$
The rest is to apply residue theorem.
5. Define
$$g(z)=f(z)\prod_{k=1}^{n}\frac{R^2-\overline{a_k}z}{R(z-a_k)}$$
Show that
   1. lg(z)l = lf(z)l when $|z|=R$.
   2. Since g has no zeroes in D_R, $\log g$ exists on D_R by question 1.
   3. Apply question 4 for u being the real part of $\log g$.